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Orientifolds of the 3-sphere

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Abstract: We study the geometry of orientifolds in the $\mathrm{SU}(2)_k$ WZW model. They correspond to the two inequivalent, orientation-reversing involutions of S^3, whose fixed-point sets are: the north and south poles ($\mathcal{O}0$), or the equator two-sphere ($\mathcal{O}2$). We show how the geometric action of these involutions leads unambiguously to the previously obtained algebraic results for the Klein bottle and Möbius amplitudes. We give a semiclassical derivation of the selection rules and signs in the crosscap couplings, paying particular attention to discrete B-fluxes. A novel observation, which does not follow from consistency of the one-loop vacuum diagrams, is that in the case of the $\mathcal{O}0$ orientifolds only integer- or only half-integer-spin Cardy states may coexist.

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1. Introduction and summary

The purpose of the present work is to study orientifolds of the Wess-Zumino-Witten model on the group manifold of SU(2). The problem has been considered from an algebraic point of view in [1]–[4]. Here we will elucidate its geometry, thereby clarifying and extending the previously obtained algebraic results. Our approach will be analogous to the one used for the D-branes of this model in [5] (for some related works see [6]–[12]). One important motivation for studying the SU(2) WZW model is its relevance in describing the near-horizon geometry of the Neveu-Schwarz fivebranes [13].[1] The WZW models are, also, of interest as toy models of warped compactifications of type-I string theory (for a discussion of this point, and more references, see [16]).

Our results in this paper can be summarized as follows:

1. The possible (classes of) orientifolds correspond to the inequivalent orientation-reversing $\mathbb{Z}_2$ isometries of S^3. The requirement of orientation reversal follows from the invariance of the Wess-Zumino term in the σ-model action. The fixed points of the two such inequivalent isometries are (a) the north and south poles, or (b) the equator two-sphere (denoted respectively $\mathcal{O}0$ and $\mathcal{O}2$).

[1] An early discussion of orientifolds of this geometry can be found in [14, 15].

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2. The action of the $\mathbb{Z}_2$ isometries on closed-string vertex operators dictates the form of the Klein-bottle amplitudes in the direct channel. One recovers the result proposed in [1, 2, 3]. The same amplitudes in the transverse channel give the couplings of the crosscap to closed-string states, up to a sign.

3. In general, perturbative orientifolds can be further distinguished by the B-flux they support [17, 18]. This flux is either integer ($\mathcal{O}^-$) or half-integer ($\mathcal{O}^+$). In our case, the two $\mathcal{O}0$'s at the S^3 poles are of the same or opposite type, according to the parity of the Kac-Moody level k. This is consistent with previous observations [19]–[22] that an $\mathcal{O}6$ orientifold changes type when intersecting a NS fivebrane. This fact, together with the explicit form of hyperspherical harmonics, explain furthermore the selection rules found in the crosscap couplings, and fix completely the sign ambiguities.

4. Combining the crosscap couplings with the well-known D-brane couplings, gives the Möbius-strip amplitudes in the closed channel. The transformation to the direct channel fixes the signs of the projections on open-string states. We explain why these are consistent with the geometric action of the $\mathbb{Z}_2$ isometries. One subtle feature, that does not follow from consistency of one-loop vacuum amplitudes alone, is that only half of the WZW branes may coexist for a given choice of $\mathcal{O}0$ orientifolds.

5. We verify that the absolute value of the crosscap couplings reduces, for large radius, to the expected tensions of flat-space orientifolds. We also extend the analysis in [5], and show that the Dirac-Born-Infeld action gives the exact (ratios of) D-brane couplings to all higher, closed-string harmonics on S^3. Finally, we discuss briefly the extension of our analysis to AdS_3.

Note added in proof: while we were completing this paper, there appeared the preprints [34, 35] which contain some overlapping results.

2. Geometric $\mathcal{O}0$ and $\mathcal{O}2$ orientifolds

The SU(2) group manifold can be parametrized as

$$g = \frac{1}{L}\begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix}, \tag{2.1}$$

with the X_i taking values on a three-sphere of radius L. Two standard parametrizations of the sphere are (a) in terms of polar coordinates:

$$X_1 = L\cos\psi\,, \qquad X_2 = L\sin\psi\cos\theta\,, \qquad X_3 + iX_4 = L\sin\psi\sin\theta\; e^{i\phi}\,, \tag{2.2}$$

with $\psi, \theta \in [0,\pi]$ and $\phi \in [0, 2\pi]$, or (b) in terms of Euler angles:

$$X_1 + iX_2 = L\sin\alpha e^{i\beta}\,, \qquad X_3 + iX_4 = L\cos\alpha e^{i\gamma}\,, \tag{2.3}$$

with $\alpha \in [0, \pi/2]$ and $\beta, \gamma \in [0, 2\pi]$. Strictly-speaking, the Euler angles are α and the two linear combinations $\beta \pm \gamma$.

The orientifold operation, Ωh, is a combination of worldsheet orientation reversal, which for closed strings reads

$$\Omega: \ \sigma \to 2\pi - \sigma \,, \qquad \text{or} \qquad z \equiv e^{\tau + i\sigma} \to \bar{z} \,, \tag{2.4}$$

and of a $\mathbb{Z}_2$ isometry h of the target manifold. All isometries of the three-sphere can be realized as $O(4)$ rotations in the embedding four-dimensional space, so that modulo conjugation by a group element

$$h = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1) \,. \tag{2.5}$$

The fact that h is an isometry ensures the invariance of the metric term in the σ-model action. However, the non-trivial Neveu-Schwarz antisymmetric background imposes one extra condition. Since Ω flips the orientation of any 3-manifold whose boundary is the string worldsheet, h must *flip the orientation of the target three-sphere* whose volume form is proportional to the NS field strength $H = dB$. This is required for the invariance of the Wess-Zumino term in the σ-model action. Put differently, we must make sure that the orientifold projection does not eliminate the B-field components that are turned on in the WZW background. We are thus left with two inequivalent possibilities (see figure 1):

- $h_0 = \mathrm{diag}(1, -1, -1, -1)$ which maps $g \to g^\dagger$. In terms of polar and of Euler coordinates this action reads:

$$(\psi, \theta, \phi) \to (\psi, \pi - \theta, \pi + \phi) \,, \qquad \text{and} \qquad (\alpha, \beta, \gamma) \to (\alpha, -\beta, \pi + \gamma) \,. \tag{2.6}$$

  The fixed points of this transformation are a pair of $\mathcal{O}0$ orientifolds, located at the north and at the south pole of the three-sphere (at $\psi = 0, \pi$).

- $h_2 = \mathrm{diag}(-1, 1, 1, 1)$ which maps $g \to -g^\dagger$, or explicitly

$$(\psi, \theta, \phi) \to (\pi - \psi, \theta, \phi) \qquad \text{and} \qquad (\alpha, \beta, \gamma) \to (\alpha, \pi - \beta, \gamma) \,. \tag{2.7}$$

  The fixed-point locus of this isometry is an $\mathcal{O}2$ orientifold that wraps the equator two-sphere (at $\psi = \pi/2$).

Notice that there are neither $\mathcal{O}1$, nor $\mathcal{O}3$ orientifolds. Note also that in a complete string background the orientifolds may acquire extra spatial dimensions, as we will discuss later in the context of the type-II NS fivebrane.

In general, the type-II orientifold may contain both Ωh_0 and Ωh_2, and hence also their product $h_0 h_2 = -1$. The latter is the freely-acting isometry that is modded out when passing from SU(2) to SO(3) $\simeq$ SU(2)/$\mathbb{Z}_2$. In this case, quantization of the NS flux requires that the Kac-Moody level k be even. The corresponding orientifolds are, in the language of [1, 2, 3], 'open descendants' of the non-diagonal D-series models. The SO(3) model in particular will be discussed in a separate paper by one of us [23]. More general orientifolds gauge a group of the form $\Gamma \oplus \Omega h_0 \Gamma$ or $\Gamma \oplus \Omega h_2 \Gamma$, with Γ any discrete group of orientation-preserving isometries. This would be relevant, in particular, for orientifolds of the E-series models, as well as for orientifolds of string compactifications on the Lenz spaces $S^3/\mathbb{Z}_n$. We will not discuss such models in the present work.

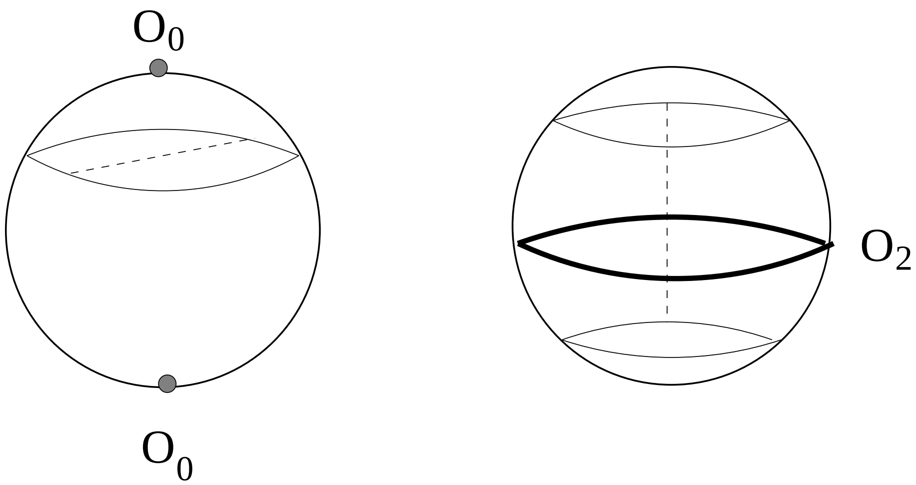

Figure 1: The two possible orientifolds for a three-sphere that is threaded by non-vanishing NS three-form flux. The thin circles are two-spheres at fixed polar angle ψ, while the broken lines connect pairs of identified points. The $\mathcal{O}1$ and $\mathcal{O}3$ orientifolds are not consistent with the background flux.

3. Klein bottle and selection rules

From the geometric actions (2.6) and (2.7) we can easily deduce the Klein bottle amplitudes. These implement the orientifold projection on closed-string states, which in the (diagonal) bosonic A-series models are of the form

$$\mathcal{P}(J_n^a, \bar{J}_{\bar{n}}^{\bar{a}})\ |j, m, \bar{m}\rangle \otimes |\mathrm{rest}\rangle\,. \tag{3.1}$$

Here $\mathcal{P}(J_n^a, \bar{J}_{\bar{n}}^{\bar{a}})$ is a polynomial in the left and right Kac-Moody currents, and $|j, m, \bar{m}\rangle$ is a lowest-weight state transforming in the (j, j) representation of the global $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The allowed values of j are $0, 1/2, \ldots, k/2$, and $m, \bar{m}$ are the J^3 and $\bar{J}^3$ eigenvalues. The full string-theory background has extra CFT components besides the WZW model — we have denoted the corresponding component of the closed-string state by $|\mathrm{rest}\rangle$. Physical states must of course also obey the Virasoro conditions.

We need the action of the orientifold operations on (3.1). First, notice that both Ωh_0 and Ωh_2 exchange the left- with the right-moving currents,

$$\Omega h:\ J = \frac{k}{2} g\partial g^{\dagger} \ \longleftrightarrow\ \bar{J} = \frac{k}{2} g^{\dagger}\bar{\partial} g\,. \tag{3.2}$$

The two inequivalent orientifolds differ, however, in their action on the primary states. The corresponding vertex operators are the (ultra)spherical harmonics, which can be expressed as homogeneous polynomials of degree $2j$ in the cartesian coordinates X_M. In the Euler parametrization they take the form

$$\mathcal{D}^j_{m\bar{m}}(\alpha, \beta, \gamma) = e^{i(m+\bar{m})\gamma}\ e^{i(m-\bar{m})\beta} P^j_{m\bar{m}}(\cos\alpha)\,, \tag{3.3}$$

where $P^j_{m\bar{m}}$ solve a second-order differential equation, and are related to the Jacobi functions [24]. The explicit form of these functions is not important for our purposes here. All that matters is that they are symmetric under the interchange of m and $\bar{m}$. Using this fact, and the transformations (2.6) and (2.7), we find the following actions on the closed-string primaries:

$$\Omega h_0 |j, m, \bar{m}\rangle = (-)^{m+\bar{m}} |j, \bar{m}, m\rangle\,, \qquad \text{and} \qquad \Omega h_2 |j, m, \bar{m}\rangle = |j, \bar{m}, m\rangle\,. \tag{3.4}$$

We are now ready to write down the Klein-bottle amplitudes. These receive contributions only from $m = \bar{m}$ states. Denoting by χ_j the Kac-Moody characters, and using the identity $2m = 2j$ mod 2, we find:

$$\mathcal{K}_{(0)} = \frac{1}{2}\,\mathrm{Tr}\left[\Omega h_0 q^{L_0+\bar{L}_0-\frac{c}{12}}\right] = \frac{1}{2}\sum_{j=0}^{k/2} (-)^{2j} \chi_j(q^2) Z_{\rm rest}(q^2)\,, \tag{3.5}$$

and

$$\mathcal{K}_{(2)} = \frac{1}{2}\,\mathrm{Tr}\left[\Omega h_2 q^{L_0+\bar{L}_0-\frac{c}{12}}\right] = \frac{1}{2}\sum_{j=0}^{k/2} \chi_j(q^2) Z_{\rm rest}(q^2)\,. \tag{3.6}$$

Here $q = e^{-2\pi t}$, and $Z_{\rm rest}$ is the contribution of the CFT with which the WZW model is being tensored. The overall sign must be chosen so as to symmetrize the states in the singlet ($j = 0$) sector. This ensures that the identity operator is not eliminated by the orientifold projection. Expressions (3.5) and (3.6) agree with those proposed, on the basis of purely algebraic arguments, by Pradisi et al. [1]. We have derived them here from a geometrical viewpoint.

To extract the tension and other properties of the orientifolds we need to express the Klein bottle amplitudes as an exchange of a closed string between crosscaps. This transformation to the 'transverse channel' is achieved by the change of variables

$$q = e^{-2\pi t} \to \tilde{q} = e^{-2\pi/t}\,, \tag{3.7}$$

and by using the modular property of the characters

$$\chi_i(q^2) = S_i^{\ j}\, \chi_j(\sqrt{\tilde{q}}\,)\,. \tag{3.8}$$

Consistency requires the final result to be of the general form

$$\mathcal{K} = \sum_{j=0}^{k/2} (C^j)^2\, \chi_j(\sqrt{\tilde{q}}\,)\; \tilde{Z}_{\rm rest}(\sqrt{\tilde{q}})\,, \tag{3.9}$$

with the C^j's giving the orientifold (or crosscap) couplings to closed strings in the (j, j) representation of the current algebra.

The modular-transformation matrix of the WZW model reads:

$$S_i^{\ j} = \sqrt{\frac{2}{k+2}}\,\sin\left(\frac{(2i+1)(2j+1)\pi}{k+2}\right)\,. \tag{3.10}$$

Putting this expression in (3.5) and (3.6), and doing the sums of the sine functions, leads to the 'crosscap coefficients' [1]:

$$C^j_{(0)} = \varepsilon^j_{(0)}\, E_{2j+k}\, \sin\left(\frac{(2j+1)\pi}{2k+4}\right) \sqrt{\mathcal{N}_j}\,, \tag{3.11}$$

and

$$C^j_{(2)} = \varepsilon^j_{(2)} E_{2j} \cos\left(\frac{(2j+1)\pi}{2k+4}\right) \sqrt{\mathcal{N}_j}\,. \tag{3.12}$$

Here E_n is a projector onto even integers (we will also need the projector onto odd integers, O_n). Explicitly:

$$E_n = \frac{1+(-)^n}{2}\,, \qquad O_n = \frac{1-(-)^n}{2}\,. \tag{3.13}$$

We have also introduced the normalization factors

$$(\mathcal{N}_j)^{-1} \equiv \sqrt{\frac{k+2}{2}} \sin\left(\frac{(2j+1)\pi}{k+2}\right), \tag{3.14}$$

for reasons that will become apparent later. Since we have only computed the squares of the C^j's, there is at this point a sign ambiguity parametrized by $\varepsilon^j_{(0)}$ and $\varepsilon^j_{(2)}$. Notice that all the crosscap couplings are real — changing the overall sign of the Klein bottle would have made them imaginary and is, hence, inconsistent.

To understand the meaning of these formulae let us focus in particular on the couplings of the dilaton. These should follow from an effective orientifold action

$$S = T_{\mathcal{O}} \int e^{\Phi} \sqrt{-\hat{g}} + \cdots\,, \tag{3.15}$$

where $T_{\mathcal{O}}$ is the orientifold tension, $\hat{g}$ the determinant of the induced metric, and the dots stand for higher α' corrections. The power of e^{Φ} is the Euler character of the real projective plane, which is the same as that of the disk diagram. Now let us expand the dilaton in terms of hyperspherical harmonics:

$$\Phi = \sum_j \sum_{L=0}^{2j} \sum_{M=-L}^{L} \Phi_{jLM}\, F_{jLM}(\psi) Y_{LM}(\theta,\phi)\,. \tag{3.16}$$

We are here using a basis in which, instead of fixing the eigenvalues of J^3 and $\bar{J}^3$ (as in the previous section), we are fixing the angular momentum under the diagonal SO(3) subgroup of SO(4) $\simeq$ SU(2) $\times$ SU(2). This is convenient because only the $L = M = 0$ components couple to spherical sources, like our orientifolds or the WZW D-branes. The other components in the decomposition (3.16) will vanish upon integration over θ and ϕ. The coupling of the jth dilaton harmonic to a spherical source located at ψ_0 must, therefore, be proportional to $F_{j00}(\psi_0)$.

Consider first the $\mathcal{O}2$ orientifold, which is located at the equator of the three-sphere, $\psi = \pi/2$. The functions $F_{j00}(\psi)$ are the Gegenbauer polynomials:

$$F_{j00}(\psi) = \sqrt{\frac{2}{\pi}} \frac{\sin(2j+1)\psi}{\sin\psi}\,, \tag{3.17}$$

while $Y_{00} = 1/\sqrt{4\pi}$. From equations (3.15) and (3.16) we thus derive the following effective couplings of $\mathcal{O}2$ to the Φ_{j00}:

$$C^j_{\text{eff }(2)} = 2\sqrt{2}\, T_{\mathcal{O}2} L^2\, (-)^j\, E_{2j}\,, \tag{3.18}$$

where L is the radius of the three-sphere, and we have used the obvious identity: $\sin[(2j+1)\pi/2] = (-)^j\, E_{2j}$ for integer or half-integer j.

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This effective field-theory calculation 'explains', firstly, the selection rule, i.e. the presence of the projector E_{2j}. Indeed, harmonics corresponding to odd $2j$ vanish on the equator two-sphere, and hence do not couple to the $\mathcal{O}2$ orientifold (at least to leading order in α'). The same calculation fixes also the sign ambiguity in expression (3.12):

$$\varepsilon^{j}_{(2)} = (-)^j \,\mathrm{sign}(T_{\mathcal{O}2})\,. \tag{3.19}$$

We will confirm these signs by computing the Möbius amplitudes later on. A more detailed comparison of (3.18) with (3.12) is a priori possible only in the semiclassical (large radius) limit, and requires a careful match of the normalizations of exchanged closed-string states. We will return to this problem in the following section. Suffice, for the time being, to say that both the selection rules, and the sign of the couplings is not affected by normalization factors.

Let us consider next the $\mathcal{O}0$ orientifolds, whose coupling to the dilaton is the sum of a north-pole and a south-pole term:

$$T^{\mathrm{north}}_{\mathcal{O}0}\Phi(\psi=0) + T^{\mathrm{south}}_{\mathcal{O}0}\Phi(\psi=\pi)\,. \tag{3.20}$$

Now the only F_{jLM} that don't vanish at the poles are those with $L = M = 0$ (or else the function Φ would have been singular). Evaluating the Gegenbauer polynomials at the poles (by taking a limit) leads to the following effective $\mathcal{O}0$ couplings:

$$C^{j}_{\mathrm{eff}\,(0)} = \frac{\sqrt{2}\,(2j+1)}{2\pi}\left[T^{\mathrm{north}}_{\mathcal{O}0} + (-)^{2j}T^{\mathrm{south}}_{\mathcal{O}0}\right]. \tag{3.21}$$

To proceed further we next need to look into the precise nature of the north and south orientifolds. Indeed, as explained very clearly in [18] (see also [17, 21, 22]), perturbative orientifolds come in two types according to the discrete torsion of the Neveu Schwarz field B.[2] More explicitly, the orientifolding operation replaces a small sphere S^n around the orientifold by one copy of $\mathbb{RP}^n$. In the case at hand, assuming for simplicity that the remaining space time coordinates are spectators, we have $n = 2$. The flux of B through this $\mathbb{RP}^2$ can be either half-integer or integer — this is the only gauge-invariant statement one can make. The corresponding orientifolds are denoted $\mathcal{O}^+$ and $\mathcal{O}^-$. These give rise, in the flat-space limit, to respectively symplectic or orthogonal D-brane gauge groups.

What kind of orientifolds do we have in our case? A priori we may think we are free to choose, but there is one global consistency condition. The B-flux out of the $\mathbb{RP}^2$ around a pole is half the flux coming out of the corresponding S^2 before the orientifold projection. Consider now two infinitesimal two-spheres around the north and the south poles of S^3. Since (in units $4\pi^2\alpha' = 1$)

$$\int_{\mathrm{north}} B - \int_{\mathrm{south}} B = \int_{S3} H = k\,, \tag{3.22}$$

[2]Torsion associated with the Ramond-Ramond fields is invisible in perturbation theory, and will not concern us here.

the two orientifolds are of the same type for k even, and of opposite type if k is odd.[3] Orientifolds of opposite type have tensions that differ by a sign, so that

$$T_{\mathcal{O}0}^{\rm north} = (-)^k \, T_{\mathcal{O}0}^{\rm south} \, . \tag{3.23}$$

Plugging into (3.21) leads to our final expression for the effective $\mathcal{O}0$ couplings:

$$C^j_{\rm eff\ (0)} = \frac{\sqrt{2}\,(2j+1)}{\pi} \, T_{\mathcal{O}0}^{\rm north} \, E_{2j+k} \, . \tag{3.24}$$

This 'explains' the selection rule found in (3.11) by the Klein-bottle calculation, and fixes also the sign ambiguity:

$$\varepsilon^j_{(0)} = {\rm sign}(T_{\mathcal{O}0}^{\rm north}) \, . \tag{3.25}$$

The Möbius strip calculation in section 5 will give an independent confirmation of these signs. First, however, we turn to a comparison of the orientifold couplings C^j with the couplings of WZW D-branes.

4. Comparing D-branes and orientifolds

The WZW SU(2) model has one type of Cardy state [25] for each integer or half-integer value of the spin s, with $0 \leq s \leq k/2$. The explicit expression of these boundary states gives directly their couplings to closed-string fields. Equivalently, one can read these couplings off the annulus diagram. For an open string stretching between a D-brane of type r and one of type s the annulus diagram reads:

$$\mathcal{A}_{rs} = {\rm Tr}\left(\sqrt{q}^{\;L_0 - c/24}\right) = \sum_{j=0}^{k/2} N_{rs}{}^j \, \chi_j(\sqrt{q}) Z_{\rm rest}(\sqrt{q}) \, . \tag{4.1}$$

Here $N_{rs}{}^j$ are the Verlinde fusion coefficients, whose non-zero values are

$$N_{rs}{}^j = 1 \, , \qquad {\rm for}\ |r-s| \ \leq j \leq \min(r+s, k-r-s) \, . \tag{4.2}$$

Using the modular transformation of the characters we can write the amplitude in the transverse (cylinder) channel as follows:

$$\mathcal{A}_{rs} = \sum_{j=0}^{k/2} D^j_r D^j_s \, \chi_j(\tilde{q}^2) \tilde{Z}_{\rm rest}(\tilde{q}^2) \, , \tag{4.3}$$

where

$$D^j_s = \sin\left(\frac{(2j+1)(2s+1)\pi}{k+2}\right) \sqrt{\mathcal{N}_j} \, , \tag{4.4}$$

and $\mathcal{N}_j$ are the normalization coefficients given by eq. (3.14). The D^j_r describe the couplings of a D-brane of type r to the closed strings in the (j,j) representation of the current algebra. They are the counterparts of the crosscap coefficients C^j of the previous section. Note that in an orientifold background they should be multiplied by an extra factor of $1/\sqrt{2}$.

[3] Note that the usual shift $k \to k+2$, that takes care of α' corrections, does not modify this conclusion since 2 is even.

The semiclassical study of the Cardy states [5] is based on the Dirac-Born-Infeld action for a D2-brane,[4]

$$S = T_{D2} \int e^{\Phi} \sqrt{-\det(\hat{g} + \hat{B} + 2\pi\alpha' F)} + \cdots . \tag{4.5}$$

Here T_{D2} is the brane tension and F the worldvolume gauge field. We consider for definiteness seven flat spectator dimensions, so that g is the natural metric on $R^7 \times S^3$. This background has ofcourse a dilaton tadpole, but this will not affect our present discussion. Let us now choose a convenient gauge in which the Neveu-Schwarz background reads

$$B = \left[L^2 \left(\psi - \frac{\sin 2\psi}{2} \right) + \pi\alpha' n_0 \right] \sin\theta \, d\theta \, d\phi \, , \tag{4.6}$$

where n_0 is an integer parametrizing a (residual) freedom of 'large' ψ-independent gauge transformations. We will say more about this freedom later on. Using the underlying group symmetry, we can furthermore bring the center-of-gravity of the branes to the north pole. The stable configurations can then be shown [5] to correspond to two-spheres spanned by the polar angles θ and ϕ, and carrying a worldvolume 'monopole' flux

$$F = -\frac{n}{2} \sin\theta \, d\theta \, d\phi \, . \tag{4.7}$$

The branes sit at the latitude

$$\psi_n = (n - n_0) \, \frac{\pi\alpha'}{L^2} \, , \tag{4.8}$$

where the DBI energy is minimized. To make contact with the CFT, one must relate the label s of the Cardy states to the (gauge-invariant) flux,

$$n - n_0 = 2s + 1 \, , \tag{4.9}$$

and identify the three-sphere radius with the Kac-Moody level, $L^2 = (k+2)\alpha'$. We have included here the well-known radius shift, which effectively resums an entire series of α' corrections to the WZW background.

Consider now the dilaton couplings, which follow by linearizing the DBI action (4.5). Plugging (4.6)–(4.8) in this action one finds

$$\sqrt{-\det(\hat{g} + \hat{B} + 2\pi\alpha' F)} = L^2 \, \sin\psi_n \, \sin\theta \, . \tag{4.10}$$

Notice the single power of $\sin\psi_n$ in this expression. Simple geometrical area is proportional to $\sin^2\psi_n$, but the DBI energy is larger because of the contribution of B and F. Expanding the dilaton in hyperspherical harmonics as in (3.16), and using the Gegenbauer polynomials (3.17), leads easily to the following effective D-brane couplings:

$$D^j_{\mathrm{eff}\ s} = 2\sqrt{2} T_{D2} \, L^2 \, \sin \left[\frac{(2j+1)(2s+1)\pi\alpha'}{L^2} \right] . \tag{4.11}$$

These should be compared to the exact CFT results (4.4).

[4]There is also a dual description in terms of a D-particle matrix model, see [9][26]–[29].

The first thing to notice is that for given j, and after taking into account the radius shift, the ratios of effective couplings match exactly those of the CFT,

$$\frac{D^j_{\text{eff } s}}{D^j_{\text{eff } r}} = \frac{D^j_s}{D^j_r}\,. \tag{4.12}$$

This agreement was pointed out for $j = 0$ in [5], and we see here that it continues to hold for all j. It suggests that the DBI and CFT calculations differ only in the precise normalization of the exchanged closed-string states. In order to compare this normalization, we must use the relation between the two different bases of hyperspherical harmonics,

$$|j, L = M = 0\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} |j, m, -m\rangle \tag{4.13}$$

The field theory calculation gives the coupling to the state on the left-hand-side, while the CFT calculation gives the coupling to each individual state $|j, m, -m\rangle$. The meaningful ratio is thus

$$\frac{1}{2j+1}\left(\frac{D^j_{\text{eff } s}}{D^j_s}\right)^2 = 4\pi\sqrt{2}\,(T_{D2}\,\alpha')^2 (k+2)^{3/2}\left[\frac{\sin x_j}{x_j}\right] \tag{4.14}$$

where

$$x_j \equiv \frac{(2j+1)\pi}{k+2}\,. \tag{4.15}$$

The j-independent factor in the right-hand-side of (4.14) is uninteresting — it simply accounts for the gravitational coupling and S^3 volume sitting in front of the 7d supergravity lagrangian [5]. The remaining factor inside the brackets approaches one in the semiclassical, $k \to \infty$, limit. For finite k it should, in principle, arise from α' corrections to the closed-string action. These combine elegantly, in the case at hand, to form the ratio of a quantum and a classical dimension. This fact could prove of interest in searching for gravitational analogs of the Born-Infeld action (see [30] for a recent discussion of non-linear gravitational actions).

Let us go back now to the orientifolds, whose closed-string couplings are given by equations (3.11) and (3.12). In section 3 we explained the origin of the selection rules, and reduced the sign ambiguities to an overall, j-independent sign. In order to better understand now the magnitude of the crosscap couplings, it is convenient to compare them to those of D-branes, whose normalization we have discussed with great care. Taking ratios one finds:

$$\frac{C^j_{(0)}}{D^j_s} = \pm E_{2j+k}\,\frac{\sin(x_j/2)}{\sin[(2s+1)x_j]}\,, \tag{4.16}$$

and

$$\frac{C^j_{(2)}}{D^j_s} = \pm E_{2j}\,\frac{\cos(x_j/2)}{\sin[(2s+1)x_j]}\,. \tag{4.17}$$

In the large-radius limit the D2-brane reduces to a collection of $2s+1$ D-particles, all sitting at the north pole of the three sphere. Using the well-known relation between D-brane tensions (see [31]) one finds:

$$\frac{C^0_{(0)}}{D^0_s} \to \pm\frac{1}{2(2s+1)}E_k \qquad \text{and} \qquad \frac{C^0_{(2)}}{D^0_s} \to \pm\frac{4\pi L^2 T_{D2}}{(2s+1)T_{D0}}\,. \tag{4.18}$$

This is what one should expect for, respectively, two $\mathcal{O}0$ orientifolds (of same or opposite type according to the parity of k), and for an $\mathcal{O}2$ orientifold wrapping the equator two sphere.[5]

It is intriguing to observe that the exact expressions of the crosscap couplings of the $\mathcal{O}0$s, resemble those for a (hypothetical) D-brane with (gauge-invariant) flux one half. This is suggestive of an exact semiclassical argument, like the one that worked for the D-branes, though there is no a priori reason why such an argument should exist.

5. Möbius amplitudes

After the Klein bottle and the annulus, we turn now our attention to the Möbius amplitude which implements the orientifold projection on open-string states. Our geometric analysis will both clarify and extend the algebraic results of [1]. We will show, in particular, that in the $\mathcal{O}0$ case only half of the WZW D-branes can coexist. This is a finer requirement, not dictated solely by consistency of the genus-one vacuum amplitudes.

In order to write down Möbius amplitudes it is convenient to work with the basis of real characters:

$$\hat{\chi}_j(q) \equiv e^{-i\pi(h_j - c/24)}\,\chi_j(-\sqrt{q})\,. \tag{5.1}$$

The Möbius amplitude is given by a linear combination of $\hat{\chi}_j(q)$ in the direct channel, and $\hat{\chi}_j(\tilde{q})$ in the transverse channel. The relevant transformation is implemented by the matrix

$$P = T^{1/2} S T^2 S T^{1/2}\,. \tag{5.2}$$

In the case of the WZW SU(2) model a straightforward calculation gives [1, 2]

$$P_i^{\ j} = \frac{2}{\sqrt{k+2}}\,\sin\left(\frac{\pi(2i+1)(2j+1)}{2(k+2)}\right) E_{2i+2j+k}\,. \tag{5.3}$$

We have included, for the reader's convenience, the calculation of this P matrix (which requires some finite Gauss sums) in an appendix.

Since we only study SO(3)-invariant configurations, our D2-branes will be always parallel to the $\mathcal{O}2$, or have the $\mathcal{O}0$s at their geometrical center. More general configurations, including branes and orientifolds at angles, are beyond our present scope. We also restrict our attention to elementary branes, and will not discuss Chan-Paton multiplicities.

[5] To check the factors of 2 in the above expressions, note for example that in the case of a flat three-torus and a single D-particle ($s=0$), a similar calculation would have given a numerical factor $2^{3/2}/\sqrt{2}$. Since on T^3 there are eight $\mathcal{O}0$s , while on S^3 there are only two, we have to divide this result by 4. This reproduces the factor in (4.18).

5.1 D-branes of $\mathcal{O}0$ models

Since h_0 maps the spherical D2-branes onto themselves, the Möbius amplitude involves in this case open strings attached to a single D-brane. The corresponding primaries are spherical harmonics with standard parity properties,

$$Y_{lm}(\pi-\theta,\phi+\pi)=(-)^l\,Y_{lm}(\theta,\phi)\,. \tag{5.4}$$

For a D-brane of type r, the Möbius amplitude in the open channel therefore reads

$$\mathcal{M}_{(0)\ r}=\varepsilon'_r\sum_{l=0}^{l_0}(-)^l\hat{\chi}_l(q)\hat{Z}_{\rm rest}(q)\,, \tag{5.5}$$

where $l_0=\min(2r,k-2r)$, and we have allowed for an arbitrary overall sign, ε'_r, which would determine the type of projection for D-brane of type r. Transforming the above amplitude to the closed channel with the P matrix, leads after some elementary algebra to:

$$\mathcal{M}_{(0)\ r}=\varepsilon'_r(-)^{2r}\sum_j|C^j_{(0)}|D^j_r\,\hat{\chi}_j(\tilde{q})\,\hat{Z}_{\rm rest}(\tilde{q})\,. \tag{5.6}$$

The crosscap and D-brane coefficients are given by (3.11) and (4.4). Consistency requires the $C^j_{(0)}$ to enter in this expression without absolute value, or extra signs. This is the case provided: (a) the sign of the crosscap coefficients does not depend on the spin j, in agreement with our semiclassical reasonning leading to (3.25) in section 3, and (b) the projection alternates with the D-brane type,

$$\varepsilon'_r=\text{sign}(T^{\rm north}_{\mathcal{O}0})\ (-)^{2r}\,. \tag{5.7}$$

Note that at the level of genus-one vacuum diagrams all types of D-branes are a priori allowed, provided the above condition is respected.

From the geometric viewpoint this conclusion cannot, however, be correct. The $l=0$ open-string states include three translation zero modes [5] that would survive the projection if ε'_r were positive. But rigid translations displace the geometric center of the brane away from the north pole of S^3, and are inconsistent with the h_0 symmetry. Thus ε'_r better be always negative. Equation (5.7) then implies that only branes of 'integer type' are in the spectrum for an $\mathcal{O}0^-$ orientifold at the north pole, while only 'half-integer types' are allowed if the north-pole orientifold is $\mathcal{O}0^+$. In either case, there are no zero modes in the open-string spectrum.

Another argument, leading to the same conclusion, goes as follows: Since the D2-branes in the orientifold theory are geometrically $\mathbb{RP}^2$s, the monopole flux (n) through them has to be even. From eq. (4.8), which relates n to the D-brane type, we conclude that only D-branes of integer (half-integer) type s are allowed when n_0 is odd (even). But $n_0/2$ is precisely the B-flux through an $\mathbb{RP}^2$ around the orientifold north-pole, as can be seen from eq. (4.6). Thus the type of the north-pole orientifold determines the set of allowed Cardy states, in agreement with our previous argument about zero modes.

5.2 D-branes in $\mathcal{O}2$ models

Let us move on now to the involution h_2, which maps a D-brane of type r onto its mirror brane of type $k/2-r$. The D2-branes will thus come in pairs, except when k is even in which case there is an invariant brane at the equator. The generic Möbius amplitude involves strings stretching between mirror branes, and transforming in Kac-Moody representations with spin $j = k/2 - 2r, \ldots, k/2$. Since the spherical harmonics do not transform under h_2, we expect the Ωh_2 projection to be j-independent.[6] Thus the Möbius amplitude in the open channel should read:

$$\mathcal{M}_{(2)\ r} = \varepsilon'_r \sum_{j=k/2-2r}^{k/2} \hat{\chi}_j(q) \hat{Z}_{\rm rest}(q)\,. \tag{5.8}$$

Transforming to the transverse channel with the P matrix, and comparing with the expected form,

$$\mathcal{M}_{(2)\ r} = \sum_j C^j_{(2)} D^j_r\ \hat{\chi}_j(\tilde{q}) \hat{Z}_{\rm rest}(\tilde{q})\,, \tag{5.9}$$

we find that $\varepsilon'_r = \mathrm{sign}(T_{\mathcal{O}2})$ for all r. All (pairs) of WZW D-branes are allowed here, and all but the equatorial brane have zero modes. These correspond to simultaneous rigid motions of a brane and of its mirror image.

Let us conclude this section with a subtle point. Naively, one may have thought that the B field should vanish on the equator two-sphere, where the geometric h_2 action is trivial. This is the case for the background

$$B = L^2 \left(\psi - \frac{\sin 2\psi}{2} - \frac{\pi}{2}\right) \sin\theta\, d\theta\, d\phi\,, \tag{5.10}$$

which corresponds to the choice $-n_0 = k/2 + 1$ in eq. (4.6). For k even this is indeed an allowed choice, but when k is odd it has (observable) singularities at the poles. Does this mean that the $\mathcal{O}2$ orientifold is inconsistent unless k is even (like the $\mathrm{SU}(2)/\mathbb{Z}_2$ orbifold)? The answer is 'no'. The field B can, in fact, be non-zero on the orientifold two-sphere, provided B and $-B$ are related by a (large) gauge transformation. When k is odd our spherical orientifold has a non-trivial $\mathbb{Z}_2$ flux on its worldvolume, which is however perfectly consistent. An analogous situation is known to arise in the flat, toroidal case [32]. Notice that the discrete B-flux on the $\mathcal{O}2$ should not be confused with the discrete fluxes out of an $\mathbb{RP}^2$ surrounding an $\mathcal{O}0$. The difference is similar to the difference between a magnetic charge and a flux line.

6. $\mathrm{SL}(2,\mathbb{R})$ and NS fivebranes

It would be interesting to extend our analysis to GKO coset models, Gepner models, and to other group manifolds. In place of concluding remarks, let us briefly discuss here the special case of the group manifold $\mathrm{SL}(2,\mathbb{R})$, and comment also on the embedding of our orientifolds in the near-horizon geometry of NS fivebranes.

[6] This is obvious for the strings on equatorial brane, from which one can then generalize to other branes by continuity.

The manifold of $\mathrm{SL}(2,\mathbb{R})$ is a hyperboloid in $\mathbb{R}^{2,2}$. This is obvious when one parametrizes real 2×2 matrices as follows:

$$g = \frac{1}{L}\begin{pmatrix} X^0+X^1 & X^2+X^3 \\ X^2-X^3 & X^0-X^1 \end{pmatrix}. \tag{6.1}$$

Cylindrical coordinates are defined through the relations

$$X^0+iX^3 = L\cosh\rho e^{i\tau}\,, \qquad X^1+iX^2 = L\sinh\rho e^{i\phi}\,. \tag{6.2}$$

The AdS_3 spacetime is the universal cover, obtained by ignoring the periodic identification of the time coordinate τ.

We are interested in orientation-reversing $\mathbb{Z}_2$ isometries. Because of the indefinite metric there exist four inequivalent such elements of $O(2,2)$, as opposed to only two for $O(4)$. They can be chosen as follows:

- $X^2 \to -X^2$, or $(\tau,\phi)\to(\tau,-\phi)$;
- $X^3 \to -X^3$, or $(\tau,\phi)\to(-\tau,\phi)$;
- $X^{1,2,3} \to -X^{1,2,3}$, or $(\tau,\phi)\to(-\tau,\pi+\phi)$;
- $X^{0,2,3} \to -X^{0,2,3}$, or $(\tau,\phi)\to(\pi+\tau,-\phi)$.

Only the first of these isometries gives a priori a 'physically-allowed' orientifold of AdS_3 spacetime. The last one is an involution of the hyperboloid, but not of its universal covering space. The second and third isometries change the orientation of time. Their fixed-point sets are spacelike surfaces, whose interpretation is unclear.

The 'physical' orientifold is an AdS_2 slice dividing the AdS_3 cylinder in two equal parts. This meshes nicely with the fact that the only 'physical' symmetric D-branes in this model have also AdS_2 geometry [33]. An interesting question is whether one can construct fully-consistent type I backgrounds as orientifolds of type IIB theory on $AdS_3{\times}S_3$. The possible orientifolds in orbifold spaces, such as the BTZ black hole, may also be interesting to consider.

One of the motivation for studying the WZW models is that they arise in the near-horizon geometry of NS fivebranes. Recall that the throat region of $N=k+2$ parallel NS5-branes is described by an exact conformal field theory [13] comprising: (i) a level-k WZW model with group SU(2) that corresponds to the 3-sphere surrounding the branes, (ii) a Feigin-Fuchs or linear-dilaton field corresponding to the radial direction, and (iii) six flat spacetime coordinates along the fivebranes. Closed type II string theory in this background is conjectured to be dual to the 5+1 dimensional worldvolume theory of the fivebranes. Since the coupling blows up at zero radius, classical string theory is, to be sure, not reliable in the entire throat region.

How are our WZW orientifolds realized in this particular context? Let us take the NS-fivebranes along the dimensions 056789, so that the three-spheres surrounding them are given by $X_1^2+X_2^2+X_3^2+X_4^2=L^2$, as in section 2. To preserve an $O(3)$ symmetry, the orientifold must intersect the branes at right angles, and extend along either one or three transverse dimensions. Assume for simplicity that the NS branes live inside the orientifold, i.e. that this latter extends also along 056789. There are then two distinct possibilities: (i) an $\mathcal{O}6$ plane cutting the three-sphere at the two poles, or (ii) an $\mathcal{O}8$ plane cutting S^3 at an equator two-sphere. These are precisely the orientifolds we found in the WZW model.

Now it is a well-known fact that when an $\mathcal{O}6$ crosses an odd number of NS-branes, it changes its type from $\mathcal{O}6^-$ to $O6^+$ and vice versa [19]–[22]. This agrees again with our observation that when the Kac-Moody level k is odd, the north- and south-pole orientifolds must be of opposite type. The topological argument for this is, as a matter of fact, the same in both contexts.

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A. Calculation of the P matrix

For the reader's convenience, we give here the computation of the P matrix [1], equation (5.3).[7]

In order to simplify the formulae, the SU(2) representations are labelled by their dimension rather than by their spin. The matrices S and T read:

$$S_{ab} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi ab}{k+2}\right) \qquad \text{and} \qquad T_{ab} = \exp\left(\frac{i\pi a^2}{2(k+2)} - \frac{i\pi}{4}\right)\delta_{ab}\,. \tag{A.1}$$

It follows easily that

$$(ST^2S)_{ab} = -\frac{i}{k+2}\left(L_{a-b} - L_{a+b}\right), \tag{A.2}$$

where

$$L_x = \sum_{c=0}^{k+1}\exp\left(\frac{i\pi c^2}{k+2}\right)\cos\left(\frac{\pi x c}{k+2}\right). \tag{A.3}$$

Now using the following Gauss sum, valid for integer x:

$$\frac{1}{2}\sum_{c=0}^{k+1}\left[\exp\left(\frac{i\pi}{k+2}(c+\frac{x}{2})^2\right) + \exp\left(\frac{i\pi}{k+2}(c-\frac{x}{2})^2\right)\right] = \\ = \exp\left(\frac{i\pi}{4}\right)\sqrt{k+2}\,E_{k+x} + \exp\left(\frac{i\pi x^2}{4(k+2)}\right)O_{k+x}\,, \tag{A.4}$$

[7] We thank Gianfranco Pradisi for kindly communicating his notes.

one finds after some algebra:

$$L_x = \exp\left(-\frac{i\pi x^2}{4(k+2)} + \frac{i\pi}{4}\right)\sqrt{k+2}\, E_{k+x} + O_{k+x}\,. \tag{A.5}$$

Noting that $E_{k+a+b} = E_{k+a-b}$ and $O_{k+a+b} = O_{k+a-b}$ leads to the equation

$$L_{a-b} - L_{a+b} = 2\sqrt{k+2}\,\exp\left(-\frac{i\pi(a^2+b^2)}{4(k+2)}\right)\sin\left(\frac{\pi ab}{2(k+2)}\right)E_{k+a+b}\,. \tag{A.6}$$

Multiplying (A.1) by $T^{1/2}$ on the left and right leads to the final expression for the P matrix, equation (5.3).

References

[1] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Planar duality in* SU(2) *WZW models*, *Phys. Lett.* **B 354** (1995) 279 [hep-th/9503207].

[2] G. Pradisi, A. Sagnotti and Y.S. Stanev, *The open descendants of nondiagonal* SU(2) *WZW models*, *Phys. Lett.* **B 356** (1995) 230 [hep-th/9506014]; *Completeness conditions for boundary operators in 2d conformal field theory*, *Phys. Lett.* **B 381** (1996) 97 [hep-th/9603097].

[3] A. Sagnotti and Y.S. Stanev, *Open descendants in conformal field theory*, *Fortsch. Phys.* **44** (1996) 585 [hep-th/9605042].

[4] J. Fuchs, L.R. Huiszoon, A.N. Schellekens, C. Schweigert and J. Walcher, *Boundaries, crosscaps and simple currents*, *Phys. Lett.* **B 495** (2000) 427 [hep-th/0007174].

[5] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[6] C. Klimcik and P. Severa, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112].

[7] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[8] S. Stanciu, *D-branes in group manifolds*, *J. High Energy Phys.* **01** (2000) 025 [hep-th/9909163].

[9] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040].

[10] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030].

[11] J. Pawelczyk, SU(2) *WZW D-branes and their noncommutative geometry from dbi action*, *J. High Energy Phys.* **08** (2000) 006 [hep-th/0003057].

[12] J. Maldacena, G.W. Moore and N. Seiberg, *D-brane charges in five-brane backgrounds*, *J. High Energy Phys.* **10** (2001) 005 [hep-th/0108152].

[13] C.G. Callan Jr., J.A. Harvey and A. Strominger, *Supersymmetric string solitons*, hep-th/9112030.

[14] S. Förste, D. Ghoshal and S. Panda, *An orientifold of the solitonic fivebrane*, *Phys. Lett.* **B 411** (1997) 46 [hep-th/9706057].

[15] M. Bianchi and Y.S. Stanev, *Open strings on the Neveu-Schwarz pentabrane*, *Nucl. Phys.* **B 523** (1998) 193 [hep-th/9711069].

[16] C. Bachas, *D-branes in some near-horizon geometries*, hep-th/0106234.

[17] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[18] E. Witten, *Baryons and branes in anti de Sitter space*, *J. High Energy Phys.* **07** (1998) 006 [hep-th/9805112].

[19] N. Evans, C.V. Johnson and A.D. Shapere, *Orientifolds, branes and duality of 4d gauge theories*, *Nucl. Phys.* **B 505** (1997) 251 [hep-th/9703210].

[20] S. Elitzur, A. Giveon, D. Kutasov and D. Tsabar, *Branes, orientifolds and chiral gauge theories*, *Nucl. Phys.* **B 524** (1998) 251 [hep-th/9801020].

[21] A. Hanany and B. Kol, *On orientifolds, discrete torsion, branes and m theory*, *J. High Energy Phys.* **06** (2000) 013 [hep-th/0003025].

[22] O. Bergman, E.G. Gimon and S. Sugimoto, *Orientifolds, RR torsion and K-theory*, *J. High Energy Phys.* **05** (2001) 047 [hep-th/0103183].

[23] N. Couchoud, *Branes and orientifolds of* SO(3), in preparation

[24] N. Vilenkin, *Special functions and the theory of group representations*, American Mathematical Society, Providence 1968.

[25] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[26] K. Hashimoto and K. Krasnov, *D-brane solutions in non-commutative gauge theory on fuzzy sphere*, *Phys. Rev.* **D 64** (2001) 046007 [hep-th/0101145].

[27] Y. Hikida, M. Nozaki and Y. Sugawara, *Formation of spherical D2-brane from multiple D0-branes*, *Nucl. Phys.* **B 617** (2001) 117 [hep-th/0101211].

[28] J. Pawelczyk and H. Steinacker, *Matrix description of D-branes on 3-spheres*, hep-th/0107265.

[29] D.P. Jatkar, G. Mandal, S.R. Wadia and K.P. Yogendran, *Matrix dynamics of fuzzy spheres*, hep-th/0110172.

[30] S. Deser and G.W. Gibbons, *Born-Infeld-einstein actions?*, *Class. and Quant. Grav.* **15** (1998) L35 [hep-th/9803049].

[31] J. Polchinski, *Tasi lectures on D-branes*, hep-th/9611050;
I.W. Taylor, *Lectures on D-branes, gauge theory and M(atrices)*, hep-th/9801182;
C.P. Bachas, *Lectures on D-branes*, hep-th/9806199.

[32] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365.

[33] C. Bachas and M. Petropoulos, *Anti-de-sitter D-branes*, *J. High Energy Phys.* **02** (2001) 025 [hep-th/0012234].

[34] I. Brunner, *On orientifolds of WZW models and their relation to geometry*, hep-th/0110219.

[35] L.R. Huiszoon, K. Schalm and A.N. Schellekens, *Geometry of WZW orientifolds*, hep-th/0110267.